PH4204: High Energy Physics

1. **Scalar propagator and causality:** Evaluate the function

$$\langle 0|\phi(x)\phi(y)|0\rangle = D(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}e^{-ip.(x-y)}$$

for $(x-y)$ spacelike so that $(x-y)^2 = -r^2$.

2. **Dirac equation and all that:** In Weyl representation the gamma matrices are given by

$$\gamma^0 = \begin{bmatrix} I_2 & 0_2 \\ 0_2 & I_2 \end{bmatrix} \text{ and } \gamma^i = \begin{bmatrix} 0_2 & \sigma^i \\ -\sigma^i & 0_2 \end{bmatrix} .$$

   (a) Show that the four matrices satisfy the Clifford algebra that is $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}I$.

   (b) Show that the matrices $\sigma^{\mu\nu}$, defined by $\sigma^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$ are block diagonal in this representation.

   (c) Write down the four component Dirac spinor as $\Psi = \begin{bmatrix} \Psi_L \\ \Psi_R \end{bmatrix}$, where Ψ_L and Ψ_R are two component matrices (called Weyl spinors). Write down the Dirac equation in terms of equations obeyed by Ψ_L and Ψ_R respectively.

   (d) Show that for $m = 0$ the Dirac equation decomposes into independent equations for Ψ_L and Ψ_R.

3. **Lorentz transformation and spinors:** Consider the transformation matrix

$$L(\theta,\phi) = \begin{bmatrix} cosh\ \phi & -sinh\ \phi & 0 & 0 \\ -sinh\ \phi & cosh\ \phi & 0 & 0 \\ 0 & 0 & cos\ \theta & -sin\ \theta \\ 0 & 0 & sin\ \theta & cos\ \theta \end{bmatrix}$$

   (a) Show that it is a Lorentz transformation matrix.

   (b) Consider the case when $\theta,\ \phi\ <<\ 1$. Write down the values of the parameters $\omega_{\rho\sigma}$ for this infinitesimal transformation. Write down the 4×4 matrix A where A is the deviation of L from the identity matrix to the first order in θ and ϕ.

   (c) Show by an explicit calculation that $L = \text{Exp}(A)$.

   (d) Find the 4×4 matrix $S(\Lambda)$ by which the Dirac spinor transforms when this finite Lorentz transformation is carried out. Carry out the calculation in the representation where

$$\gamma^0 = \begin{bmatrix} I_2 & 0_2 \\ 0_2 & -I_2 \end{bmatrix} \text{ and } \gamma^i = \begin{bmatrix} 0_2 & \sigma^i \\ \sigma^i & 0_2 \end{bmatrix} .$$

   (e) By an explicit calculation show that $S(\Lambda)$ obeys $S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^\mu_\nu\gamma^\nu$.

4. **Time evolution:** Consider the Hamiltonian composed of two parts $H = H_0 + H_{int.}$, where H_0 is the unperturbed Hamiltonian and $H_{int.}$ is the addition to it due to some interaction. Define the state vector and the operator as follows

$$|\psi(t)\rangle = e^{iH_0t}\,|\psi(t)\rangle_s \qquad\qquad \hat{O}(t) = e^{iH_0t}\hat{O}_s e^{-iH_0t}$$

where 's' in the suffix refers the Schrodinger picture.

(a) Find the time evolution equation for the state vector and the operator.

(b) In the interaction picture if we define the time evolution operator in the following way

$$|\Psi(t)\rangle = U(t,t_0)\,|\Psi(t_0)\rangle$$

where the $U(t,t_0)$ has the explicit form

$$U(t,t_0) = e^{iH_0 t} e^{-iH(t-t_0)} e^{-iH_0 t_0}$$

then find out its time evolution equation. Also show that $U(t,t_0)$ can be written down as

$$U(t,t_0) = T\left(e^{-i\int_{t_0}^{t} dt' H_{int.}(t')}\right)$$

where T denotes the time ordering operator.

5. **Green's functions:** The retarded propagator (two point Green's function) for a real scalar field $\Phi(x)$ is defined as

$$D_R(x) = \theta(x^0)\,\langle 0|\,[\hat{\Phi}(x),\hat{\Phi}(0)]\,|0\rangle \ .$$

Consider this propagator in the Heisenberg picture for a free field theory where the field operator is given by

$$\hat{\Phi}(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega}}\left[\hat{a}e^{-i\omega t+i\vec{p}.\vec{x}} + \hat{a}^\dagger e^{i\omega t-i\vec{p}.\vec{x}}\right]$$

where $\omega = \sqrt{p^2+m^2} \geq 0$.

(a) Starting from the definition of retarded propagator $D_R(x)$ show this may be written as

$$D_R(x) = \int \frac{d^4p}{(2\pi)^4}\frac{i}{(p_0-i\epsilon)^2-\omega^2}e^{-ip_0t+i\vec{p}.\vec{x}} \ .$$

The integrals are from $-\infty$ to $+\infty$ along the real axis and ϵ is a real, positive but infinitesimally small quantity.

(b) Show that $(\partial^2+m^2)D_R(x) = -i\delta^4(x)$.

6. **Wick's theorem and Feynman diagrams:**

(a) Write down the Wick's theorem for bosonic fields along with the contraction rules.

(b) Draw the Feynman diagrams for the self-interacting scalar fields with $\lambda\Phi^4$ interaction. And explain the cancellation of the bubble diagrams in any Green's function calculation.

(c) Also find a solution to the 6-fermion Wick's theorem,

$$\langle 0|\,T\{\Psi(x_1)\Psi(x_2)\Psi(x_3)\bar{\Psi}(x_4)\bar{\Psi}(x_5)\bar{\Psi}(x_6)\}\,|0\rangle$$

where $\Psi(x)$'s are the fermionic fields.

7. **Yukawa theory and more:** Consider a theory with real scalar field ϕ of mass m and Dirac field ψ with mass M with interaction Lagrangian given by

$$\mathcal{L}_{int.} = \frac{\lambda}{4!}\phi^4 + \frac{\kappa}{2}\bar{\psi}\psi\,\bar{\psi}\psi + g\,\bar{\psi}\psi\,\phi \ .$$

where the coupling constants λ, κ and g are real and a measure of interaction strength.

(a) Write down the Feynman rules for writing matrix element for this theory.

(b) Consider the scattering process $\psi\psi \to \psi\psi$. Draw the Feynman diagrams (without any loops) for the process and write down the expressions for those diagrams.